Eigenstructure Assignment by the Differential Sylvester Equation for Linear Time-Varying Systems

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This work is concerned with the assignment of the desired eigenstructure for linear time-varying systems such as missiles, rockets, fighters, etc. Despite its well-known limitations, gain scheduling control continues to be a major focus of research efforts. Scheduling of frozen-time, frozen-state controllers for fast time-varying dynamics is known to be mathematically fallacious and practically hazardous. Therefore, recent research efforts are being directed towards applying time-varying controllers. In this paper, we i) introduce a differential algebraic eigenvalue theory for linear time-varying systems, and ii) propose an eigenstructure assignment scheme for linear time-varying systems via the differential Sylvester equation based upon newly developed notions. The whole design procedure of the proposed eigenstructure assignment scheme is very systematic. The scheme can be used to determine the stability of linear time-varying systems easily as well as to provide a new horizon of designing controllers for linear time-varying systems. The presented method is illustrated by a numerical example.

Key Words : Differential Sylvester Equation, Eigenstructure Assignment, Linear Time-Varying Systems

1. Introduction

In the study of the flight control of ordinary aircraft, the pertinent equations of motion have coefficients dependent on flight speed. Previously, it has been conventional practice to consider the speed as a constant, resulting in linear differential equations with constant coefficients. The relatively small accelerations experienced by subsonic aircraft rendered the assumption a reasonable one in that good results were obtained. With much increased accelerations and velocities of modern supersonic aircraft and missiles, however, the parameters dependent on flight velocity change at a significantly rapid rate. Further, a high rate of fuel consumption causes the mass, center of gravity, and moment of inertia of a vehicle to alter significantly during the characteristic response time of the controlled motions. In addition, variations of flight conditions in rapid ascent through the atmosphere introduce time-varying parameter variations. Consequently, modern automatic flight control analysis of aircraft and missiles requires the inclusion of these time-varying parameters such that the responses are characterized by differential equations with variable coefficients.

Eigenvalues and eigenvectors of a linear transformation (of its matrix representation) play very important roles in the analysis of linear time-invariant dynamical systems such as the determination of system stability. However, it is well known that the eigenvalues of a linear time-varying system matrix $A(t)$ do not determine the stability of the linear time-varying system. It is

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also known that an algebraic transformation $x(t) = T(t)\bar{x}(t)$ with $T(t)$ formed by the eigenvectors of $A(t)$ will not, in general, result in a simpler form (such as the diagonal or the Jordan form) for $\bar{A}(t) = T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t)$ in the transformed equivalent system, nor will it preserve the invariancy of its eigenvalues.

For years, many researchers have attempted to generalize the conventional notions of eigenvalues and eigenvectors for linear time-invariant systems to linear time-varying systems. Wu(1974) proposed the extended-eigenvalue (X-eigenvalue) and extended-eigenvector (X-eigenvector) notion, which are defined as a scalar function $\lambda(t)$ and a time-varying vector function $u(t)$, respectively, in the equation $[A(t) - \lambda(t)I]u(t) = \dot{u}(t)$. While this extension is a great leap from the generally invalid frozen-time extension, it is not very useful because, by the well-known existence theorem for the solution of linear time-varying systems, any scalar function $\lambda(t)$ is an X-eigenvalue for any matrix $A(t)$. This means that $\lambda(t)$ is not uniquely determined for linear time-varying systems. Therefore, the very essence of being "eigen-" is lost in the attempt to extend eigenvalue concepts. The results of Richards(1983) about the Floquet characteristic exponent were fundamental to the understanding of performance and stability for linear periodic time-varying systems. Nemytskii and Stepanov (1960) further studied the Lyapunov characteristic exponent while Kamen(1988) developed notions on poles and zeros for linear time-varying systems, and Zhu and Morales(1992) introduced a notion of co-eigenvalue. Zhu and Johnson(1991) developed a new time-dependent eigenvalue(SD/PD-eigenvalue) theory and an associated set of matrix canonical forms for a matrix over a differential ring using the differential algebraic structure and a classical result on differential operator factorization developed by Cauchy and Floquet. Tsakalis and Ioannou (1993) extended the pole placement control objective to linear time-varying plants.

On the one hand, the problem of eigenstructure assignment(simultaneous assignment of eigenvalues and eigenvectors) is of great importance in control theory and applications because the stability and dynamic behavior of a linear time-invariant multivariable system are governed by the eigenstructure of the system. In general, the speed of a response is determined by the assigned eigenvalues whereas the shape of the response is furnished by the assigned eigenvectors. Eigenstructure assignment is an excellent method for incorporating classical specifications on damping, settling time, and mode decoupling into a modern multivariable control framework. The eigenstructure assignment algorithm can be divided into two groups: the right eigenstructure (eigenvalues/right eigenvectors) assignment, and the left eigenstructure(eigenvalues/ left eigenvectors) assignment. Their roles in designing a control system are distinctly different(Choi, 1998a, 1998b; Choi *et al.*, 1995a, 1995b, 1996).

In this paper, we introduce the notion of eigenstructure(eigenvalue/eigenvector) for linear time-varying systems proposed by Zhu *et al.* (1999), and propose an eigenstructure assignment scheme for linear time-varying systems via the differential Sylvester equation. The notion of Eigenstructure for linear time-varying systems introduced in this paper is a generalized notion of existing concepts for linear time-invariant systems, and the eigenstructure assignment scheme proposed in this paper is also a generalized algorithm that includes existing time-invariant cases.

The remainder of this paper is organized as follows. In Sec. 2, we briefly introduce a newly developed eigenstructure notion, and define a control design objective obtained by assigning the desired closed-loop eigenvalues and eigenvectors appropriately for linear time-varying systems using the novel notion of eigenstructure. In Sec. 3, we review an eigenstructure assignment scheme for linear time-invariant systems via the Sylvester Equation, and we also propose an extended eigenstructure assignment scheme for linear time-varying systems via the differential Sylvester equation. The presented method is illustrated by a numerical example in Sec. 4, and finally conclusions and suggestions for future works are stated in Sec. 5.

2. Eigenstructure for Linear Time-Varying Systems

2.1 Preliminaries

Some technical preliminaries for defining the eigenstructure of linear time-varying systems are presented in this section. In order to obtain the eigenstructure of linear time-varying systems, we introduce a unified spectral theory for Nth-order scalar linear time-varying systems of the form:

$$y^{(N)}(t)+\alpha_N(t)y^{(N-1)}(t)+\cdots+\alpha_2(t)\cdot\dot{y}(t)+\alpha_1(t)y(t)=0$$
$$y^{(k)}(t_0)=y_{k0},\ k=0,1,,\cdots,N-1 \quad (1)$$

Eq. (1) can be conveniently represented as $D_\alpha\{y\}=0$ using the scalar polynomial differential operator (SPDO)

$$D_\alpha=\delta^N+\alpha_N(t)\delta^{N-1}+\cdots+\alpha_2(t)\delta+\alpha_1(t) \quad (2)$$

where $\delta=d/dt$ is a derivative operator. It is well-known that the subclass of linear time-invariant systems Eq. (1), where $\alpha_k(t)\equiv\alpha_k$, enjoys a spectral theory that facilitates analytical solutions, precise stability criteria, frequency domain analysis and synthesis, and stabilization control design techniques. However, as is also well-known, this spectral theory for time-invariant systems does not carry over, in general, to the time-varying case in a transparent manner. Recently, a unified spectral theory has been developed for linear time-varying systems given in Eq. (1), based on a classical result of Floquet (1883) on the factorization of SPDO

$$D_\alpha=(\delta-\lambda_N(t))\cdots(\delta-\lambda_2(t))(\delta-\lambda_1(t)) \quad (3)$$

Let $I\subseteq R$ be a real interval and let $G=G(R)$ (or $G=G(C)$) be the differential ring (D-ring) of regulated C^∞ function $f=I\to R$ (or $f=I\to C$) with $\delta=d/dt$ the derivative operator defined on K. Then the SPDO D_α defined in Eq. (2) with $\alpha_k\in K$ is an operator over the D-ring K. The basic terminology for the unified eigenvalue concept can be summarized as follows.

Definition 2. 1 (Zhu *et al.*, 1999)

(a) Let D_α be an SPDO with $\alpha_k\in K$, $k=1,2,\cdots,N$. Then, the scalar functions $\lambda_k\in K$, $k=1,2,\cdots,N$, given by the factorization Eq. (3) are called Series D-eigenvalue (SD-eigenvalue) of D_α. Moreover, $\rho(t)=\lambda_1(t)$ is called a Parallel D-eigenvalue (PD-eigenvalue) of D_α.

(b) A multi-set $\Gamma_\alpha=\{\lambda_k(t)\}_{k=1}^N$ is called a Series D-spectrum (SD-spectrum) for D_α if the $\lambda_k(t)$ satisfy Eq. (3).

(c) A set $\Upsilon_\alpha=\{\rho_k(t)\}_{k=1}^N$ is called a Parallel D-spectrum (PD-spectrum) for D_α if the $\rho_k(t)$ are PD-eigenvalues for D_α and $\{y_k(t)=\exp(\int\rho_k(t)dt)\}_{k=1}^N$ constitutes a fundamental set of solutions to $D_\alpha\{y\}=0$.

(d) Let $A_c(t)$ be the companion matrix associated with D_α:

$$A_c(t)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\alpha_1(t) & -\alpha_2(t) & \cdots & \cdots & -\alpha_N(t) \end{bmatrix} \quad (4)$$

The matrix

$$\Gamma(t)=\begin{bmatrix} \lambda_1(t) & 1 & 0 & \cdots & 0 \\ 0 & \lambda_2(t) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_N(t) \end{bmatrix} \quad (5)$$

is called a Series Spectral canonical form (SS-canonical form) for D_α and $A_c(t)$. The diagonal matrix

$$\Upsilon(t)=\mathrm{diag}[\rho_1(t),\rho_2(t),\cdots,\rho_N(t)] \quad (6)$$

is called a Parallel Spectral canonical form (PS canonical form) for D_α and $A_c(t)$.

(e) Let D_α be an Nth-order SPDO and let $\{y_i(t)\}_{i=1}^N$ be any fundamental set of solutions to $D_\alpha\{y\}=0$. Let

$$W=\begin{bmatrix} y_1 & y_2 & \cdots & y_N \\ \dot{y}_1 & \dot{y}_2 & \cdots & \dot{y}_N \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(N-1)} & y_2^{(N-1)} & \cdots & y_N^{(N-1)} \end{bmatrix} \quad (7)$$

be the Wronskian matrix associated with $\{y_i\}_{i=1}^N$. Denote by D the diagonal matrix

$$D=\mathrm{diag}[y_1,y_2,\cdots,y_N] \quad (8)$$

Then

$$WD^{-1} = V(\rho_1, \rho_2, \cdots, \rho_N) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ D_{\rho_1}\{1\} & D_{\rho_2}\{1\} & \cdots & D_{\rho_N}\{1\} \\ D^2_{\rho_1}\{1\} & D^2_{\rho_2}\{1\} & \cdots & D^2_{\rho_N}\{1\} \\ \vdots & & \ddots & \vdots \\ D^{N-1}_{\rho_1}\{1\} & \cdots & \cdots & D^{N-1}_{\rho_N}\{1\} \end{bmatrix} \quad (9)$$

where $D_{\rho_i} = (\delta + \rho_i)$, $D^k_{\rho_i} = D_{\rho_i} D^{k-1}_{\rho_i}$. The canonical coordinate transformation $V(t)$ is called the modal canonical matrix for D_a associated with the PD-spectrum $\{\rho_i\}_{i=1}^N$.

The column vectors $v_i(t)$ of $V(t)$ satisfying

$$A_c(t)v_i(t) - \rho_i(t)v_i(t) = \dot{v}_i(t) \quad (10)$$

and row vectors $u_i^T(t)$ of $U(t) = V^{-1}(t)$ satisfying

$$u_i^T(t)A_c(t) - \rho_i(t)u_i^T(t) = -\dot{u}_i^T(t) \quad (11)$$

are called right PD-eigenvectors and left PD-eigenvectors, respectively, of D_a associated with $\rho_i(t)$. SD-eigenvectors can be also defined similarly.

2.2 Modal decomposition of state-space equations for linear time-varying systems

The response of a system due to a control input with zero initial condition can be represented by using right and left PD-modal matrices of the linear time-varying system. The performance for linear time-varying systems can be obtained by appropriately assigning the closed-loop PD-eigenvalues and PD-eigenvectors of linear time-varying systems like the linear time-invariant case.

The response due to a control input u(t) for a linear time-varying system characterized by the equations

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \quad (12)$$

can be represented by (D'angelo, 1970)

$$y(t) = C(t)\left\{W(t)W^{-1}(t_0)x_0 + \int_{t_0}^{t} W(t) \cdot W^{-1}(\tau)B(\tau)u(\tau)\,d\tau\right\} \quad (13)$$

where $W(t)$ is the Wronskian matrix obtained from a fundamental set of the solutions of the homogeneous system. The response of the given system with zero initial conditions is represented by using the PD-eigenstructure described in section 2.1 and the relation $W(t)W^{-1}(t_0) = V(t)Y(t, t_0)U(t_0)$ as follows (Zhu, 1996):

$$\begin{aligned} y(t) &= C(t)\int_{t_0}^{t} W(t)W^{-1}(\tau)B(\tau)u(\tau)\,d\tau \\ &= C(t)V(t)\int_{t_0}^{t} Y(t,\tau)U(\tau)B(\tau)u(\tau)\,d\tau \\ &= \sum_{k=1}^{l}\sum_{i=1}^{N} c_k(t)v_i(t)\sum_{j=1}^{m}\int_{t_0}^{t} e^{\int_{\tau}^{t}\rho_i(\tau^*)d\tau^*} \cdot u_i^T(\tau)b_j(\tau)u(\tau)\,d\tau \end{aligned} \quad (14)$$

where $v_i(t)$ and $u_i^T(t)$ are the right and left PD-eigenvector associated with PD-eigenvalue $\rho_i(t)$, respectively, and $Y(t, t_0) = \mathrm{diag}\{e^{\int_{t_0}^{t}\rho_1(\tau^*)d\tau^*}, e^{\int_{t_0}^{t}\rho_2(\tau^*)d\tau^*}, \cdots, e^{\int_{t_0}^{t}\rho_N(\tau^*)d\tau^*}\}$. Note from Eq. (14) that the desired performance can be obtained by appropriately assigning the closed-loop PD-eigenvalues and PD-eigenvectors of the linear time-varying system. Note also from the equation that the entries of the matrix $U(t)B(t)$ provide information about the controllability of the modes from the inputs, and the entries of the matrix $C(t)V(t)$ provide information about the observability of the modes in the outputs by the PBH eigenvector test (Kailath, 1980) of linear time-varying systems.

3. Eigenstructure Assignment for Linear Time-Varying Systems via the Differential Sylvester Equation

3.1 Linear time-invariant case

An algorithm for state feedback pole assignment using the Sylvester equation was introduced in (Bhattacharyya and deSouza, 1982) and used in (Cavin III and Bhattacharyya, 1983) to have low eigenvalue sensitivity for the closed-loop system, and the problem of eigenstructure assignment via the Sylvester equation has been treated by several authors. Keel and Bhattacharyya (1985) described a procedure for the design of a dynamic compensator that stabilizes the closed-loop system and causes the closed-loop system eigenstructure to be robust in the sense of making the eigenvector set maximally orthonormal. The authors extended the algorithm introduced in

(Bhattacharyya and deSouza, 1982), and (Cavin Ⅲ and Bhattacharyya, 1983) to the output feedback case and placed eigenvalues in a region of the complex plane. Tsui(1987) summarized the existing solutions to Sylvester equation, and also presented an attractive analytical and restriction-free solution with explicit freedom. Duan(1993) proposed two new simpler solutions to Sylvester equation, and presented a complete parametric approach for right eigenstructure assignment in linear systems via state feedback based on his proposed solutions. Syrmos and Lewis(1993) solved the problem of eigenstructure assignment by output feedback by using two coupled Sylvester equations. In (Kim and Kum, 1993), the authors introduced an iterative right eigenstructure assignment via Sylvester equation to design a small gain controller. A homotopy concept was adopted to develop the scheme. Syrmos and Lewis(1994) also presented necessary and sufficient conditions in terms of a bilinear Sylvester equation for stabilizing and assigning a desired eigenstructure assignment by output feedback. Choi(1998) proposed a direct left eigenstructure assignment scheme for liner time-invariant systems via Sylvester equation.

Consider a linear time-invariant multivariable controllable system to briefly describe the existing eigenstructure assignment scheme via Sylvester equation

$$\dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + \sum_{k=1}^{m} b_k u_k(t) \quad (15)$$

$$u(t) = Kx(t) \quad (16)$$

where, (i) $x \in R^N$, $u \in R^m$ denote the state and control input, respectively; (ii) A, B, and K are real constant matrices of appropriate dimensions; and (iii) rank $B = m \neq 0$ ($m \leq N$).

If a constant real state feedback(Eq. (16)) is applied to Eq. (15), the closed-loop system becomes

$$\dot{x}(t) = (A+BK)x(t) \quad (17)$$

and the corresponding eigenvalue problem is defined by

$$(A+BK)\phi_i = \lambda_i \phi_i \quad (18)$$

where ϕ_i is the right eigenvector corresponding to the eigenvalue λ_i.

The central constraint imposed in the eigenvalue assignment problem is to determine the gain matrix K that results in a prescribed set of eigenvalues. Note that K is an $(m \times N)$ dimensional matrix; it is evident that the problem is underdetermined, and therefore for controllable systems, an infinite number of choices of gain matrices exists for given eigenvalue locations. We can choose $(N \times (m-1))$ parameters arbitrarily for N prescribed eigenvalues.

The pole placement algorithm proposed in (Cavin Ⅲ and Bhattacharyya, 1983) introduces the parameter vector $h_i \in C^m$ defined by

$$h_i = K\phi_i \quad (19)$$

Then Eq. (18) is put in the form of Sylvester equation:

$$(A - \lambda_i I)\phi_i = -Bh_i \quad (20)$$

or, in a matrix form, Eq. (20) is a generalized Lyapunov equation known as Sylvester equation

$$A\Phi - \Phi\Lambda = -BH \quad (21)$$

where $\Phi = [\phi_1, \phi_2, \cdots, \phi_N]$, $\Lambda = \text{diag}[\lambda_1, \lambda_2, \cdots, \lambda_N]$, and $H = [h_1, h_2, \cdots, h_N]$.

The pole placement scheme based on Sylvester equation(Eq. (20) or (21)) can be summarized as follows(Junkins and Kim, 1993): For given set of A, B matrices, and for a prescribed Λ matrix, we can choose a parameter matrix H and solve for Φ from Eq. (21). Then, we can solve for K from the linear system(which is simply the matrix version of Eq. (19))

$$K\Phi = H \quad (22)$$

In essence, the advantage of "guessing Λ and H" instead of "guessing K" is that the exact prescribed eigenvalue positions are guaranteed if we specify Λ and choose an appropriate H. The H matrix generates (through the solution of Eq. (21) for Λ specified) all achievable eigenvector matrices.

Note, from inversion of Eq. (20), that the closed-loop eigenvectors corresponding to given λ_i and h_i are simply

$$\phi_i = -(A - \lambda_i I)^{-1} Bh_i \quad (23)$$

Thus, if the closed-loop eigenvalues(λ_i) are distinct from their open-loop positions, the columns of H directly generate all possible correrponding closed-loop eigenvectors.

In the case of right eigenstructure assignment, unfortunately, an arbitrary choice for the complex H matrix does not usually generate an attractive set of closed-loop eigenvectors; occasionally the resulting eigenvectors are so poorly conditioned that computation of an accurate gain matrix K from Eq. (22) is not possible.

Since an arbitrary selection of H is not appropriate, it is appropriate to consider choices which have a high probability of generating attractive gain matrices. An attractive algorithm results if we seek the H matrix which makes the closed-loop modal matrix lie as close as possible to a prescribed, well-conditioned matrix. Notice that, if we select some target set of well-conditioned closed-loop eigenvectors

$$\hat{\Phi}=[\hat{\phi}_1,\ \hat{\phi}_2,\ \cdots,\ \hat{\phi}_N] \tag{24}$$

which is much easier than choosing an H matrix, considering the physical information supplied by Φ, then, we can use Eq. (20), or equivalently Eq. (21), to solve for the $\hat{H}$ that most nearly (in the least squares sense) produces the desired eigenvectors $\hat{\Phi}$. Upon substituting this solution for the H matrix and re-solving Eq. (21) for the admissible eigenvector matrix Φ, we will find $\Phi \neq \hat{\Phi}$, with the degree of approximation being problem-dependent. The resulting Φ matrix lies as close to $\hat{\Phi}$ as possible (in the least squares sense) and is typically well conditioned. The gain matrix K calculated from the solution of Eq. (22) with Φ and $\hat{H}$ will, however, place the eigenvalues exactly to within arithmetic errors.

3.2 Linear time-varying case

Consider a linear time-varying system for eigenstructure assignment

$$\begin{aligned}\dot{x}(t) &= A(t)x(t)+B(t)u(t)\\ &= A(t)x(t)+\sum_{k=1}^{m} b_k(t)u_k(t)\end{aligned} \tag{25}$$

$$u(t)=K(t)x(t) \tag{26}$$

where, (i) $x\in R^N$, $u\in R^m$ denote the state and control input, respectively; (ii) $A(t)$, $B(t)$ are sufficiently smooth functions of time which are bounded, with continuous derivatives up to order $(N-1)$, (iii) $\{A(t),\ B(t)\}$ is completely controllable for allowable parameter values; (iv) rank $B(t)=m\neq 0$, $m\leq N$.

First, we take a Lyapunov transformation for Eq. (25) to define the eigenvalue problem of linear time-varying systems, and apply state feedback to the transformed system triple $(Ac(t),\ \bar{B}(t),\ \bar{K}(t))$. The closed-loop system then becomes

$$\dot{z}(t)=(A_c(t)+\bar{B}(t)\bar{K}(t))z(t) \tag{27}$$

where $A_c(t)$ is the companion matrix of $A(t)$. The problem of transforming a linear time-varying system to the companion canonical form (phase-variable form) is considered in (Wolovich, 1968; Silverman, 1966; Seal and Stubberud, 1969; Ramar and Ramaswami, 1969), and using Eqs. (10), (11) in Sec. 2.1, we define the corresponding PD-eigenvalue problem for the transformed system as follows:

$$(A_c(t)+\bar{B}(t)\bar{K}(t))v_i(t)-\rho_i(t)v_i(t) = \dot{v}_i(t)\ :\ \text{right} \tag{28}$$

$$u_i^T(t)(A_c(t)+\bar{B}(t)\bar{K}(t))-\rho_i(t)u_i^T(t) = -\dot{u}_i^T(t)\ :\ \text{left} \tag{29}$$

where $v_i(t)$ and $u_i^T(t)$ are the right and left PD-eigenvectors, respectively, corresponding to the PD-eigenvalue $\rho_i(t)$.

The objective in this paper is to find the feedback gain matrix $K(t)$ for linear time-varying systems that the closed-loop PD-eigenvalues are achieved exactly and the desired right PD-eigenvectors are assigned to the best possible set of PD-eigenvectors, at least, in the least squares sense.

The closed-loop eigenvalue problem for the transformed linear time-varying system can be described by the following differential Sylvester equation if we use the previously defined parameter vector $h_i(t)=\bar{K}(t)v_i(t)$ in Eq. (19)

$$A_c(t)v_i(t)-\rho_i(t)v_i(t)+\bar{B}(t)h_i(t) = \dot{v}_i(t) \tag{30}$$

The matrix form of the differential Sylvester equation can be rewritten as

$$A_c(t)V(t)-V(t)\Gamma(t)+\bar{B}(t)H(t)$$

$$= \dot{V}(t) \quad (31)$$

where, $V(t) = [v_1(t), v_2(t), \cdots, v_N(t)]$, $\Upsilon(t) = \text{diag}[\rho_1(t), \rho_2(t), \cdots, \rho_N(t)]$, and $H(t) = [h_1(t), h_2(t), \cdots, h_N(t)]$. Since an arbitrary selection of $H(t)$ for the desired PD-eigenvalues matrix $\Upsilon(t)$ is not appropriate, because it does not usually generate an attractive set of closed-loop eigenvectors as stated in Sec. 3.1, we select a target set of desired closed-loop right PD-eigenvectors

$$V_d(t) = [v_{d_1}(t), v_{d_2}(t), \cdots, v_{d_N}(t)] \quad (32)$$

Then, we can use Eq. (31) to solve for the $\tilde{H}(t)$ that most nearly (in the least squares sense) produces the desired right PD-eigenvectors $V_d(t)$ as follows:

$$\tilde{H}(t) = -\bar{B}^{+}(t)(A_c(t)V_d(t) - V_d(t) \cdot \Upsilon(t) - \dot{V}_d(t)) \quad (33)$$

where $\bar{B}^{\dagger}(t)$ denotes the pseudo-inverse of the matrix $\bar{B}(t)$. Upon substituting this solution for the $H(t)$ matrix and re-solving Eq. (31) for the admissible right PD-eigenvector matrix $V_a(t)$, we will find that $V_a(t) \neq V_d(t)$ because the pseudo-inverse of a matrix is incorporated in the computation. The resulting $V_a(t)$ matrix is as close to $V_d(t)$ as possible (in the least squares sense), and exactly satisfies the following equation:

$$A_c(t)V_a(t) - V_a(t)\Upsilon(t) + \bar{B}(t)\tilde{H}(t) = \dot{V}_a(t) \quad (34)$$

The gain matrix $\bar{K}(t)$ can be calculated from the solution of $\tilde{H}(t) = \bar{K}(t)V_a(t)$ with $V_a(t)$ and $\tilde{H}(t)$. The gain matrix will, however, place the PD-eigenvalues exactly, within arithematic errors.

From the above facts, we obtain the following resulting algorithm of eigenstructure assignment for linear time-varying systems:

Algorithm:

- Step 1: Take a Lyapunov transformation to transform a linear time-varying system to the companion canonical form (phase-variable form).
- Step 2: Choose the desired closed-loop PD-spectrum $\Upsilon(t)$ and the desired right PD-modal matrix $V_d(t)$ for the transformed system.
- Step 3: Calculate the parameter matrix $\tilde{H}(t)$ as follows:

$$\tilde{H}(t) = -\bar{B}^{\dagger}(t)(A_c(t)V_d(t) - V_d(t)\Upsilon(t) - \dot{V}_d(t))$$

  where $\bar{B}^{\dagger}(t)$ denotes the psuedo-inverse of the matrix $\bar{B}(t)$.
- Step 4: Solve the differential Sylvester equation with the parameter matrix $\tilde{H}(t)$ in Step 3 to get the achieved right PD-modal matrix $V_a(t)$. That is, we solve the following differential Sylvester equation for $V_a(t)$:

$$A_c(t)V_a(t) - V_a(t)\Upsilon(t) + \bar{B}(t)\tilde{H}(t) = \dot{V}_a(t)$$

- Step 5: Calculate the feedback gain matrix $\bar{K}(t)$ for the transformed system as follows:

$$\bar{K}(t) = \tilde{H}(t)V_a^{-1}(t)$$

- Step 6: Since $\bar{K}(t)$ is obtained for the transformed system, we take the inverse Lyapunov transformation to calculate the feedback gain matrix $K(t)$ for the original system.

Remarks:

1) In Step 2 of the algorithm, to decouple the ith output from the jth mode of the closed-loop system, we select the desired right PD-eigenvector such that the ith row vector $c_i(t)$ of $C(t)$ is orthogonal to the jth column PD-eigenvector $v_j(t)$, *i.e.*,

$$(A_c(t) + \bar{B}(t)\bar{K}(t))v_j(t) - \rho_j(t)v_j(t) = 0,$$

and

$$c_i^T(t)v_j(t) = 0, \quad v_j(t) \neq 0$$

2) In Step 4 of the algorithm, the problem of obtaining explicit expressions of solutions of the differential Sylvester equation is treated in the following section.

3) In the algorithm, if the rank of the control input matrix $B(t)$ is equal to the rank of the system matrix $A(t)$, the desired right PD-eigenvectors as well as the desired PD-eigenvalues can be exactly achieved.

4) The algorithm is an extended version of the existing right eigenstructure assignment scheme via Sylvester equation for linear time-invariant

systems.

3.3 Explicit solutions of the differential Sylvester matrix equation

This section is concerned with the problem of obtaining explicit expressions of solutions of the differential Sylvester matrix equations given in Eq. (34).

Definition 3.1 (Söderström and Stoica, 1989)

Let A be an $(m\times n)$ matrix and B an $(k\times s)$ matrix. Then the Kronecker product of A and B written $A\otimes B$ is an $(mk\times ns)$ matrix defined in block form as

$$A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \vdots & a_{mn}B \end{bmatrix}$$

Definition 3.2 (Söderström and Stoica, 1989)

Let A be an $(m\times n)$ matrix, and let a_i denote its ith column:

$$A=(a_i \cdots a_n)$$

Then the $(mn\times 1)$ column vector $vec(A)$ is defined as

$$vec(A)=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

The function $vec(A)$ is said to be the *vec-function* of A.

Lemma 3.1 (Söderström and Stoica, 1989)

Let A, B, and C be matrices of compatible dimension. Then

$$vec(ABC)=(C^T\otimes A)\,vec(B)$$

Using Definitions 3.1, 3.2 and Lemma 3.1, the differential Sylvester matrix equation given in Eq. (34) can be rewritten as

$$\dot{\hat{V}}_a(t)=(I\otimes A_c(t))\,\hat{V}_a(t)+(\Upsilon(t)\cdot\otimes(-I))\,\hat{V}_a(t)+\hat{D}(t) \quad (35)$$

where $D(t)$ denotes $B(t)\tilde{H}(t)$.

Eq. (35) has the same form as the extended linear system $\dot{Y}=MY+F$. Therefore, the only solution of the differential Sylvester matrix equation is given by

$$\hat{V}_a(t)=\Phi(t,t_0)\left\{\hat{V}_a(t_0)+\int_0^t\Phi(t_0,\tau)\hat{D}(\tau)\,d\tau\right\} \quad (36)$$

where $\Phi(\cdot,\cdot)$ denotes the state transition matrix of the linear system.

4. A Numerical Example

Consider a second order two-input continuous controllable linear time-varying system

$$\dot{x}(t)=A(t)x(t)+B(t)u(t)=\begin{bmatrix} 0 & 1 \\ 2-t^2 & 2t \end{bmatrix}x(t)+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}u(t),$$

$$y(t)=C(t)x(t)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}x(t)$$

The PD-eigenvalues and right PD-eigenvectors of the open-loop system are obtained as

$$\rho_{open}(t)=\{t-1,\ t+1\},$$

$$V_{open}(t)=\begin{bmatrix} 1 & 1 \\ t-1 & t+1 \end{bmatrix}.$$

Let a set of the desired PD-eigenvalues be $\Upsilon_d(t)=\text{diag}[-t-1,\ -t-2]$ for a closed-loop system to be stable by Extended-Mean Criterion (Zhu, 1996), and let the desired right PD-modal matrix $V_d(t)$ be

$$V_d(t)=\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

to decouple the output from each mode of the closed-loop system.

Then, the parameter matrix $H(t)$ is obtained as

$$H(t)=\begin{bmatrix} t+1 & 2 \\ 2-t^2 & -3t-2 \end{bmatrix}$$

The feedback gain matrix $K(t)$ is calculated using $H(t)$ and $V_d(t)$ as

$$K(t)=\begin{bmatrix} -t-1 & -1 \\ t^2-2 & -3t-2 \end{bmatrix},$$

and the resulting closed-loop system is obtained as

$$\dot{x}(t)=(A(t)+B(t)K(t))x(t)=\begin{bmatrix} -t-1 & 0 \\ 0 & -t-2 \end{bmatrix}x(t).$$

Since the rank of the control input matrix $B(t)$

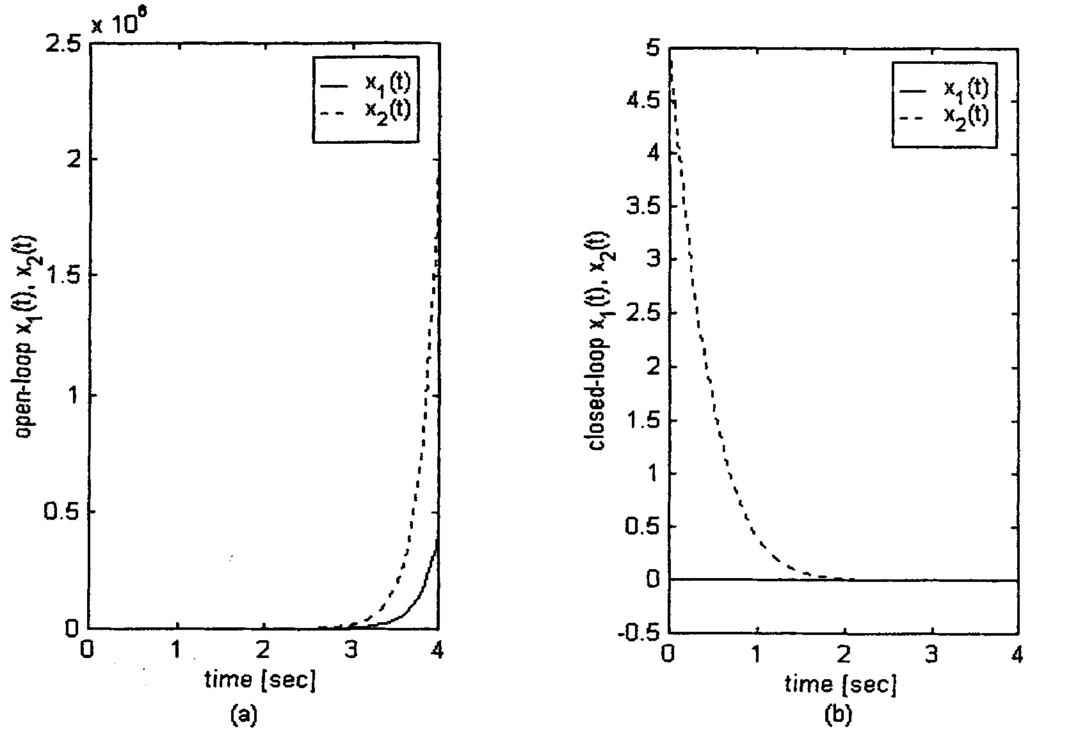

Fig. 1 Zero-input response of open-loop and closed-loop system with initial states $x_1(0)=0$, and $x_2(0)=5$.

is equal to the rank of system matrix $A(t)$ in this example, the desired right PD-eigenvectors as well as the desired PD-eigenvalues can be exactly achieved.

The zero-input responses of the open-loop and closed-loop systems with initial states of $x_1(0)=0$, $x_2(0)=5$ are depicted in Fig. 1. Figure 1(a) shows that the open-loop system is unstable. Figure 1(b) shows that the closed-loop system can be stabilized by assigning PD-eigenvalues appropriately, and the output can be decoupled from each mode by suitably assigning right PD-eigenvectors.

5. Conclusions

In this paper, we introduced new notions of eigenvalue, and proposed an eigenstructure assignment scheme for linear time-varying systems via the differential Sylvester equation that is based on these newly developed notions. We showed that closed-loop systems can be stabilized by assigning PD-eigenvalues appropriately, and the desired performance can be obtained by assigning right PD-eigenvectors according to design specifications. The algorithm proposed in this paper includes existing results on linear time-invariant system. The explicit solution of the differential Sylvester matrix equation is also presented in this paper.

The proposed eigenstructure assignment scheme via the differential Sylvester equation guarantees that the desired PD-eigenvalues are achieved exactly and the desired right PD-eigenvectors are assigned to the best possible (achievable) set of PD-eigenvectors in the least squares sense. If the number of independent actuators is equal to the dimension of a given system, the desired PD-eigenvectors as well as the desired eigenvalues can be exactly achieved. A numerical example has confirmed the usefulness of the proposed eigenstructure assignment scheme for linear time-varying systems via the differential Sylvester equation.

Some topics for future works include the application of the proposed algorithm to actual linear time-varying systems such as missiles, and to show the usefulness of the proposed algorithm.

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